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SOME PHYSICAL SOLUTIONS OF THE GENERAL EQUATION OF THE N^{TH} DEGREE

BY ROBERT E. MORITZ

SUCCESSFUL attempts to solve general equations by physical means date from about 1770. The various devices employed by Plato, Apollonius, Eratosthenes and others in the solution of the Delian problem did indeed afford solutions of the equation $x^3 = 2a^3$, but as is well known the Greeks never rose to a full conception of the general equation of even the third degree.

In 1770 J. Rowning exhibited his Universal Constructor of Equations, constructed on perfectly general principles though the working model was limited to the solution of quadratics. During the century following, L. Lalanne (1840) succeeded in constructing algebraic balances for the solution of equations of as high a degree as the seventh. Since 1870 a great number of devices for the solution of the general equation of the n^{th} degree have been invented, based upon principles of mechanics, optics, hydrodynamics, electricity and magnetism. A brief enumeration of these contrivances and their bibliography is given by R. Mehmke in the *Encyklopädie der Mathematischen Wissenschaften*, vol. I, p. 1067

The object of the present paper is to present with clearness and in an elementary way several physical solutions which, it seems to the author, deserve to be more widely known, not only because of their intrinsic interest but especially because of their pedagogic value. They offer a remarkable illustration of the inter-relation of the various elementary branches of mathematical science in revealing how various principles may lend themselves to the solution of the same problem. At the same time these solutions exhibit well the unifying effect of the mathematics, in that the most diverse physical principles are seen to conform to the same mathematical laws. Above all they form a good example of the correlation of all science, of how a comparatively difficult problem may find its complete solution in apparently unrelated fields. These elements, together with the peculiar fascination of the subject, the simplicity of the physical laws involved and the elegance of the underlying mathematical principles, commend these solutions to a place in our elementary text books.

While the ideas underlying the solutions as here presented are not new, the treatment is independent. The detailed description of the various devices has been omitted, for it is needless to say that the interest in these practical devices is essentially theoretical. Mechanical difficulties prevent the actual construction of working models for all but the lowest degrees. The best models as yet constructed give at best but a rough approximation to the roots sought. Besides it may well be questioned, whether since the perfection of approximation methods, such as Horner's, there exists any longer the need for actual solutions by physical means. For this reason the mathematical and physical principles of these devices have been emphasized rather than their mechanical execution.

The first solution is Rowning's,* based upon Segner's† method of evaluating polynomials. The second is due to Lille‡ and has been treated with modifications at some length by Cremona.§ The third solution is based upon Boys'‖ as modified by Grant.¶ The fourth, last in point of discovery, is Meslin's** hydrostatic solution. The fifth solution is that of Lucas,†† though the principle underlying it dates from Gauss.‡‡

1. A kinematic method. It is evident that if a mechanism could be devised for tracing the graph of the polynomial

$$y_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdot\cdot\cdot + a_{n-1}x + a_n \qquad [1]$$

in rectangular coordinates, the distances from the origin to the points where this graph crosses the x-axis would give at once the real roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdot\cdot\cdot + a_{n-1}x + a_n = 0\cdot \qquad [2]$$

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* *London Transactions* 1770.

† *Acad. Petropolis Novi Comment.* 1758–59.

‡ *Comptes Rendus,* 1867, p. 854.

§ *Graphical Statics,* Oxford, 1890.

‖ *Philosophic Magazine,* 1886, p. 241.

¶ *American Machinist,* 1896, p. 824.

** *Comptes Rendus,* 1900, p. 888.

†† *Comptes Rendus,* 1890, p. 965.

‡‡ *Werke,* vol. 3, p. 112.

The possibility of such a mechanism is made manifest from the following method of constructing the value of y_n in [1] for any given real value of x.

FIG. 1.

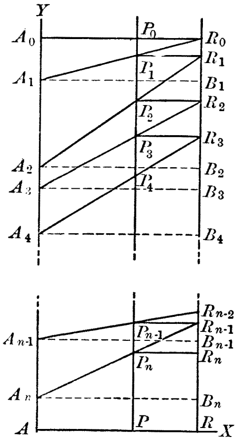


FIG. 2.

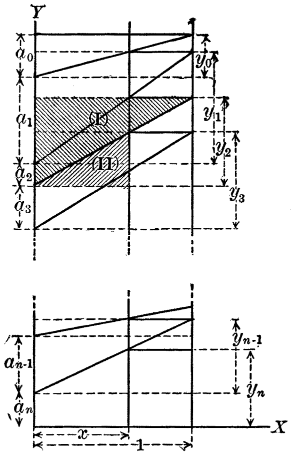
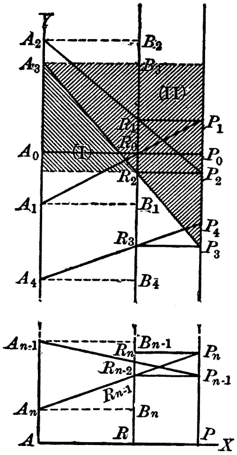


FIG. 3.



Draw the line AX (Figure 1) and on it lay off AR equal to unity and AP equal to the given value of x. Through A draw AY perpendicular to AX, and through P and R draw parallels to AY.

On AY take

$$AA_n = a_n, \quad A_nA_{n-1} = a_{n-1}, \cdots, \quad A_2A_1 = a_1, \quad A_1A_0 = a_0,$$

measuring each distance in the positive or negative direction according as the sign of the corresponding coefficient is positive or negative. Through A_0 draw a line parallel to AX, intersecting the parallels through P and R in P_0 and R_0 respectively.

Draw the line R_0A_1 and let P_1 mark its point of intersection with the line PP_0.

Draw P_1R_1 through P_1 parallel to AX intersecting RR_0 in R_1. Draw R_1A_2 and let its point of intersection with PP_0 be P_2.

Similarly determine in succession the points $P_3, P_4, \cdots, P_{n-1}, P_n$.

The distance PP_n represents the polynomial y_n.

Figure 1 gives the construction for the case in which the coefficients are all positive and $x < 1$. Figure 3 shows the construction when some of the coefficients are negative and $x > 1$.

The proof of the construction follows most readily from Figure 2, which is a repetition of Figure 1, with a more convenient notation. Through the several points A_1, A_2, A_3, $\cdots$, A_{n-1}, A_n, draw lines parallel to AX, intersecting RR_0 in the point B_1, B_2, B_3, $\cdots$, B_{n-1}, B_n respectively. Moreover let $y_0 = B_1R_0$, $y_1 = B_2R_1$, $\cdots$, $y_{n-1} = B_nR_{n-1}$, $y_n = RR_n = PP_n$.

Now we need only to compare any pair of similar triangles such as those marked (*I*) and (*II*) in Figure 2 or 3 to see that for every such pair of triangles there subsists a relation such as

$$y_2 : 1 = (y_3 - a_3) : x$$

from which

$$y_3 = y_2x + a_3.$$

We thus obtain the set of equations

$$\left.\begin{aligned} y_0 &= a_0, \\ y_1 &= y_0x + a_1, \\ y_2 &= y_1x + a_2, \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ y_{n-1} &= y_{n-2}x + a_{n-1}, \\ y_n &= y_{n-1}x + a_n, \end{aligned}\right\} \qquad [3]$$

from which by successive elimination

$$\begin{aligned} y_1 &= a_0x + a_1, \\ y_2 &= a_0x^2 + a_1x + a_2, \\ y_3 &= a_0x^2 + a_1x^2 + a_2x + a_3, \\ &\cdot \qquad \cdot \qquad \cdot \\ &\cdot \qquad \cdot \qquad \cdot \\ y_n &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n. \end{aligned}$$

This last result may also be obtained otherwise by eliminating the y's in [3] simultaneously as follows. We multiply the first of the equations [3] by x^n, the second by x^{n-1}, and similarly each equation whose first member is y_k by x^{n-k}, and then add the results. The left member of each equation except the last cancels against the first term of the right member of the equation just following and we have remaining

$$y_n = PP_n = a_0x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_{n-1}x \quad a_n.$$

From the foregoing construction a device for tracing the graph of [1] readily suggests itself. AY, AX and RR_0 are replaced by fixed rods, of which AY carries a graduated scale for the proper determination of the points $A_n, A_{n-1}, \cdots, A_1, A_0$. R_0P_0, R_1P_1, etc., R_0A_1, R_1A_2, etc., and PP_0 are replaced by adjustable rods properly grooved and pivoted so as to admit of parallel motion of the rod PP_0. While this rod PP_0 is made to move parallel to itself a pencil point at P_n traces a graph on a sheet of paper below. The points of intersection of this graph with the line AX furnish the required roots.

2. A solution by visual means. A second solution is suggested by the following method of constructing the polynomial [1].

Let us first assume all the coefficients positive.

FIG. 4.

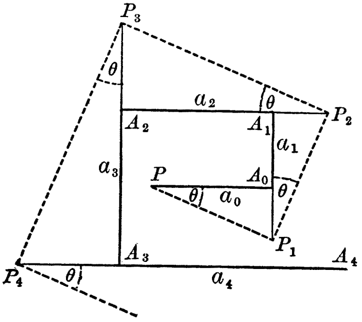
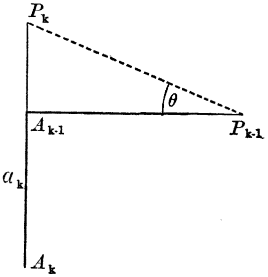


FIG. 5.



Draw $PA_0 = a_0$ (Figure 4), and from A_0 in a direction + 90° with PA_0 draw $A_0 A_1 = a_1$.

From A_1 in a direction + 90° with A_0A_1 draw $A_1A_2 = a_2$.

Similarly draw the remaining segments $a_3, \cdots, a_n$ of the broken line $PA_0A_1 \cdots A_n$.

Beginning at P draw another rectilinear figure $PP_1P_2 \cdots P_n$ having its vertices $P_1, P_2, \cdots, P_n$ respectively on the prolongations of the segments a_1 $a_2, \cdots, a_n$, and the angles at $P_1, P_2, \cdots, P_{n-1}$ right angles.

Let angle $P_1PA_0 = \theta$; then each of the angles $P_2P_1A_1$, $P_3P_2A_2, \cdots$, $P_nP_{n-1}A_{n-1}$ will also equal θ.

Moreover let

$$PA_0 = y_0 = a_0,\ P_1A_1 = y_1,\ P_2A_2 = y_2,\ \cdots,\ P_nA_n = y_n$$

Consider now any one of the similar triangles A_0PP_1, $A_1P_1P_2$, $\cdots$, $A_{n-1}P_{n-1}P_n$, say the triangle $A_{k-1}P_{k-1}P_k$ (Figure 5). We have

$$\tan\theta = \frac{P_kA_{k-1}}{P_{k-1}A_{k-1}} = \frac{P_kA_k - A_{k-1}A_k}{P_{k-1}A_{k-1}} = \frac{y_k - a_k}{y_{k-1}},$$

or, putting $\tan\theta = x$,

$$y_k = y_{k-1}\,x + a_k.$$

Hence, giving to k successively the values $0, 1, 2, \cdots, n$, we obtain equations [3] of §1, for which we have already shown that

$$y_n = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

In order, therefore, to evaluate this polynomial for a given value of x, say $x = a$, it is only necessary to take $\theta = \tan^{-1} a$, and perform the construction indicated. P_nA_n will represent the corresponding value of the polynomial.

If the coefficients a_0, a_1, $\cdots$, a_n are not all positive, the construction is modified in that the angle between consecutive segments a_{k+1} and a_k, of the line $PA_0A_1 \cdots A_n$ is taken equal to $-90°$ whenever a_{k+1} and a_k have unlike signs, to $+90°$ when they have like signs.

The reason for this rule lies in the fact that when we change the sign of the middle one of any three segments a_{k-1}, a_k, a_{k+1}, and the sense of the turn which determines its direction, keeping the absolute directions of the segments a_{k-1} and a_{k+1} unchanged, the direction of the segment a_k with reference to each of the segments a_{k-1} and a_{k+1} will have been reversed, so that the change of a single sign is accompanied by two changed turns in the line $PA_0A_1 \cdots A_n$.

Finally the sign of $P_nA_n = y_n$ is determined by the sign of $A_{n-1}A_n = a_n$, and y_n has the same or opposite sign as a_n according as P_nA_n has the same or opposite sense as $A_{n-1}A_n$.

If a is a root of [1], P_nA_n must equal zero. Now $a = \tan\theta$. We shall therefore know the roots of [1], if we can find the values of θ for which P_n coincides with A_n.

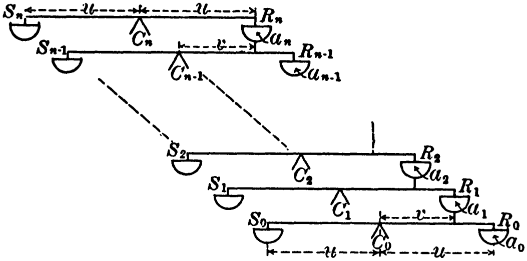
These values of θ can be found in various ways. The simplest way is to draw the line $PA_0A_1 \cdots A_n$ on a piece of tracing paper, prolonging the lines A_0A_1, A_1A_2, $\cdots$, $A_{n-1}A_n$ indefinitely in both directions. The tracing is then placed over a sheet of millimeter coordinate paper so that PA_0 will coincide with one of the ruled lines. This line is taken for the line PP_0 and the angle θ is zero. If now we gradually turn the tracing paper in a positive direction about the point P, increasing the angle θ from zero, it is evident that all positions will be reached for which P_n and A_n coincide. After some experience these positions can readily be determined visually.

3. A dynamic solution. This solution makes use of the principle of moments. Suppose we have a set of balances (Figure 6):

$$S_0R_0,\quad S_1R_1,\quad S_2R_2,\ \cdots,\ S_{n-1}R_{n-1},\quad S_nR_n,$$

with fulcrums at the middle points C_0, C_1, C_2, $\cdots$, C_{n-1}, C_n respectively. These scale beams are arranged in a vertical plane, successively one above and to the left of the other, so that each right hand scale beam, except the lowest, may be connected at a distance $+v$ from the fulcrum to the right-hand scale pan of the beam immediately above it. Let the scale beams be of equal length $2u$.

FIG. 6.



Suppose now that weights

$$a_0,\ a_1,\ a_2,\ \cdots,\ a_{n-1},\ a_n$$

are placed in the scale pans at

$$R_0,\ R_1,\ R_2,\ \cdots,R_{n-1},\ R_n$$

respectively, and that y_k denotes the resultant force at any point R_k due to the weight a_k and the weights a_{k-1}, a_{k-2}, $\cdots$, a_1, a_0 in the scale pans of the beams below it.

Then

$$y_{k+1} = X + a_{k+1}, \tag{4}$$

where X is the force resulting at R_{k+1} from the action of y_k at R . The distances of R_{k+1} and R_k from C_k are v and u respectively; hence by the principle of moments $vX = y_k u$,

or $$X = y_k x, \qquad [5]$$

where $$x = \frac{u}{v}\,.$$

Equation [4] therefore becomes

$$y_{k+1} = y_k x + a_{k+1}. \qquad [6]$$

Giving to k successively the values 0, 1, 2, $\cdots$, $n-2$, $n-1$, and remembering that $y_0 = a_0$, we obtain once more the equations [3] of § 1, from which equation [1] follows as before.

If the weight a_{k+1} had been placed in the left pan, at S_{k+1}, instead of in the right pan at R_{k+1}, equation [4] would obviously be replaced by

$$y_{k+1} = X - a_{k+1},$$

so that for every weight placed in a left pan, the corresponding coefficient in equations [3], and consequently also in equation [1], is negative.

Now since y_n denotes the resultant force effective at R_n, the following dynamic method of evaluating any polynomial of the form [1], for any real value of x, suggests itself.

We adjust our system of scale beams, n in number, by varying v so that $\frac{u}{v} = x$. We then place in one of the pans of each of the beams S_0R_0, S_1R_1, $\cdots$, S_nR_n weights respectively proportional to the coefficients a_0, a_1, $\cdots$, a_n of the polynomial, in the right or left pan according as the corresponding coefficient is positive or negative. The counterweight required to balance the beam S_nR_n represents the value of the polynomial for the given value of x. y_n is positive or negative according as the counterweight must be placed at S_n or at R_n in order to balance the system.

To determine the roots of the equation [2], the scale beams must be connected so that the system may be balanced by changing the distance v instead of by the use of counterweights. This is accomplished by attaching the fulcrums C_1, C_2, $\cdots$, C_n to parallel slides, each moved by a separate screw. These screws are geared together in such a way that as C_1 moves through a certain distance, C_2 moves through twice that distance, C_n through n times that distance. At the same time each beam slides through the ring by which the scale pan at the R above it is attached to it. Thus however the distance v may be altered, it will have the same value for each beam.

The pans having been properly weighted, the distance v is gradually varied until the system of scales is in equilibrium. For each v that this occurs the equation

$$x = \frac{u}{v}$$

furnishes a root of the equation [2].

Theoretically v may have any real value negative as well as positive. But for practical reasons v is positive and at most equal to unity, hence x is necessarily positive and not less than unity. But since the roots of

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots \pm a_{n-1}\,x \mp a_n = 0 \qquad [7]$$

are the same as those of [2] with the signs changed, and those of

$$a_nx^n + a_{n-1}\,x^{n-1} + a_{n-2}\,x^{n-2} + \cdots + a_1x + a_0 = 0 \qquad [8]$$

the reciprocals of the roots of equation [2], the positive roots of [7] with the sign changed will give the negative roots of [2] and the reciprocals of the roots of [8] whose absolute values are less than unity, will yield those roots of [2] whose absolute values are greater than unity.

4. A solution by hydrodynamics. In the preceding solution the balance was brought into equilibrium by changing the lengths of the lever-arms at which the weights act. Obviously equilibrium might also be established without changing the distances by a gradual variation of the weights.

Referring again to the polynomial [1], suppose that at the distances a_0, a_1, a_n, $\cdots$, a_{n-1} from the fulcrum C of a scale beam (Figure 7), to the left or right of C according as the coefficient is positive or negative, the respective weights

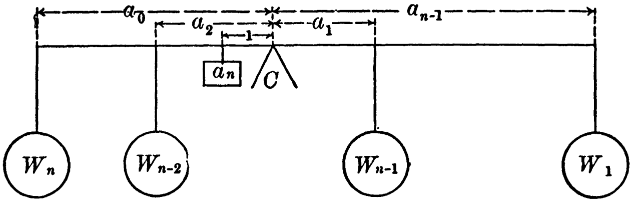
$$W_n, W_{n-1}, W_{n-2}, \cdots, W_1$$

be attached, and at a distance unity from the fulcrum a weight a_n, to the left or right according as a_n is positive or negative. The moment of rotation y_n of the system in a positive direction about C will then be

$$a_0 W_n + a_1 W_{n-1} + a_2\, W_{n-2} + \cdots + a_{n-1}\, W_1 + a_n. \qquad [9]$$

Suppose now that each weight W_k consists of a vessel containing water and that the shape of the vessel is such that the volume, and consequently the weight of its contents, varies as the k^{th} power of the height to which it is filled.

FIG. 7.



Disregarding the weights of the empty vessels and supposing that they are all filled to a common height x we should have

$$W_n = cx^n, W_{n-1} = cx^{n-1}, \cdots, W_2 = cx^2, W_1 = cx, \qquad [10]$$

where c is an arbitrary factor of proportionality which may be so chosen that the height of the water in the vessels is expressed in the same unit as the distances from c at which the weights are attached. If the weights are in grams and the distances in centimeters, c must be put equal to unity, since the weight of a given quantity of water in grams is equal to its volume in cubic centimeters. With $c = 1$ the values [10], substituted in [9], give us the polynomial [1].

If therefore we could construct vessels having the assumed forms, the value of the polynomial [1], for a positive value x of the variable, might be found from the counterweight necessary to balance the system after the vessels have been filled with water to a common height x. If the units used are grams and centimeters, the counterweight at the unit distance will give y_n directly, and it is positive or negative according as the counterweight is required at the right or the left of the fulcrum.

Now it is easy to show that vessels of the assumed form can actually be constructed. In fact for every value of k there exists a solid of revolution which satisfies the required condition that its volume equals the k^{th} power of its altitude.

For supposing that such a solid of revolution exists, take its axis for the x-axis, a tangent at its vertex for the y-axis; and let $y = f(x)$ represent the equation of a meridian of the solid. Putting V for the volume cut off from the solid by a plane perpendicular to the x-axis at a distance x, we have on the one hand, by the principles of the calculus,

$$V = \int_0^x \pi y^2 dx,$$

and on the other hand, by our assumption regarding the form of the vessel,

$$V = x^k;$$

whence

$$x^k = \int_0^x \pi y^2 dx.$$

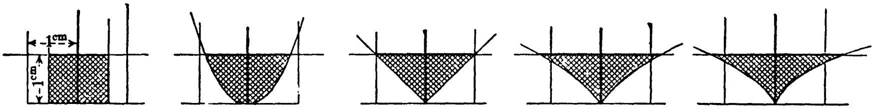
Differentiating this last equation with respect to x we obtain $kx^{k-1} = \pi y^2$, or

$$y = \sqrt{\frac{k}{\pi}} \cdot x^{\frac{k-1}{2}} . \qquad [11]$$

For $k = 1, 2, 3, 4, 5$, we have respectively $y = 0.564, (0.798)x^{\frac{1}{2}}, (0.977)x, (1.128)x^{\frac{2}{3}}, (1.262)x^{\frac{2}{23}}$.

The meridian sections of the corresponding vessels are shown in Figure 8.

FIG. 8.



$k=1$ $k=2$ $k=3$ $k=4$ $k=5$

The values of x for which the moment of rotation y_n is zero, that is, for which the system is in equilibrium without the use of counterweights, give the roots of the equation [2].

These values could be found by gradually increasing the height of the water in the vessels, keeping it at the same height in all the vessels, and observing the successive heights for which the system balances. The difficulty of keeping the height exactly the same in all the vessels is overcome by replacing the vessels by heavy solids of revolution of the same shapes and immersing these in water to a common depth. By the principle of Archimedes the upward force of the displaced water is exactly equal to the former downward force of the filled vessels. But the direction of the forces having been reversed, the weight a_n must now be attached at the opposite side of the fulcrum, that is at a unit distance to the right of C if the coefficient a_n is positive, to the left if negative. The gradual immersion of all the bodies to an equal depth is easily effected by placing the apparatus in a vessel into which water may be slowly admitted. The bodies are adjusted so that their lowest points lie in the same horizontal plane. The effect of the weights of the bodies is eliminated by carefully balancing the system before attaching the weight a_n. Then the weight a_n is attached and water admitted to the vessel. The successive depths of immersion at which equilibrium takes place give the positive roots of equation [2]. To find the negative roots we first transform the equation into another whose roots are equal to the roots of the given equation with contrary signs. The positive roots of the transformed equation will, of course, give the negative roots of the original equation.

5. An electromagnetic solution. The preceding methods have dealt with real roots of equations only; the present method furnishes all the roots of an equation. We shall therefore write the polynomial in the form

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n \qquad [12]$$

and seek the values of

$$z = x + iy$$

which will render $f(z) = 0$.

Take $n + 1$ arbitrary constants $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}$, form the function

$$F(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)(z - \lambda_{n+1}),$$

and decompose the rational fraction $f(z)/F(z)$ into partial fractions. We get

$$\frac{f(z)}{F(z)} = \frac{M_1}{z - \lambda_1} + \frac{M_2}{z - \lambda_2} + \cdots + \frac{M_n}{z - \lambda_n} + \frac{M_{n+1}}{z - \lambda_{n+1}}, \qquad [13]$$

where

$$M_1, M_2, \cdots, M_n, M_{n+1}$$

are constants given by the relation

$$M_k = \frac{f(\lambda_k)}{F'(\lambda_k)},$$

$F'(z)$ denoting as usual the derivative of $F(z)$ with respect to z.

We next take some non-conductor of electricity, say a pane of glass or a sheet of paper, to represent a plane of complex numbers; on it we select a convenient line as an x-axis, and along this line, from a point O as origin, measure the distances OR_1, OR_2, $\cdots$, OR_n, OR_{n+1} equal respectively to λ_1, λ_2, $\cdots$, λ_n, λ_{n+1}. At the points R_1, R_2, $\cdots$, R_n, R_{n+1} let the horizontal plane or pane be traversed at right angles by electric currents whose strengths are proportioned to M_1, M_2, $\cdots$, M_n, M_{n+1} respectively, with their directions downward for positive M's, upward for negative M's.

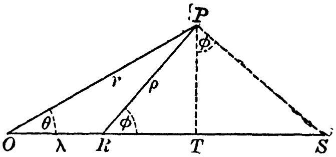
The relative current strengths are easily obtained by dividing a single circuit into branches by means of wires of uniform thickness whose various lengths are inversely proportional to the M's. The strengths of the currents carried by these branches vary inversely as the resistances (Ohm's Law), the resistances vary directly as the lengths of the wires, and the lengths of the wires vary inversely as the quantities M; consequently the current-strengths vary directly as the quantities M.

Let

$$z = x + iy$$

represent any point P in our pane (Figure 9).

FIG. 9.



By Biot and Savart's law the force exerted upon a small magnet at P by a current of indefinite length and intensity M, descending perpendicularly at R, is directly proportional to M and inversely to ρ, the distance between R and P. Its direction is PS, the normal to RP. Hence denoting this force by F, we may write

$$F = \frac{kM}{\rho}, \qquad [14]$$

where k is the factor of proportionality.

Let ϕ denote the angle which the line RP makes with the x-axis; then the angle which PS makes with PT, the perpendicular from P to the x-axis, is also ϕ. The components X, Y of the form F in the positive directions of the coordinate axes are therefore respectively

$$X = F \sin\phi, \qquad Y = -F\cos\phi,$$

whence

$$X = \frac{kMy}{\rho^2}, \qquad Y = \frac{-kM(x-\lambda)}{\rho^2}, \qquad [15]$$

since in the triangle RPT

$$\sin\phi = \frac{TP}{RP} = \frac{y}{\rho}, \qquad \cos\phi = \frac{RT}{RP} = \frac{x-\lambda}{\rho}.$$

Each of the currents at $R_1, R_2, \cdots, R_n, R_{n+1}$ gives rise to x and y components of the form [15]. Hence the resultant force at P in the direction of the x-axis is given by the sum

$$\Sigma \frac{kMy}{\rho^2} = k\Sigma \frac{My}{\rho^2} = k\left[\frac{M_1 y}{\rho_1^2} + \frac{M_2 y}{\rho_2^2} + \cdots + \frac{M_n y}{\rho_n} + \frac{M_{n+1} y}{\rho_{n+1}}\right],$$

and the resultant in the direction of the y-axis by

$$\Sigma \frac{-kM(x-\lambda)}{\rho^2} = -k\,\Sigma \frac{M(x-\lambda)}{\rho^2}$$

$$= -k\left[\frac{M_1(x-\lambda_1)}{\rho_1^2} + \frac{M_2(x-\lambda_2)}{\rho_2^2} + \cdots + \frac{M_n(x-\lambda_n)}{\rho_n^2} + \frac{M_{n+1}(x-\lambda_{n+1})}{\rho_{n+1}^2}\right].$$

At the neutral points of the electric field we have

$$\Sigma \frac{My}{\rho^2} = 0, \qquad \Sigma \frac{M(x-\lambda)}{\rho^2} = 0, \qquad (16)$$

and therefore also

$$\Sigma \frac{M(x-\lambda)}{\rho^2} - i\,\Sigma \frac{My}{\rho^2} = \Sigma\left(M\frac{x-\lambda-iy}{\rho^2}\right) = 0.$$

But

$$\rho^2 = (x-\lambda)^2 + y^2 = (x-\lambda+iy)(x-\lambda-iy),$$

whence

$$\Sigma\left(M\frac{x-\lambda-iy}{\rho^2}\right) = \Sigma \frac{M}{x-\lambda+iy} = \Sigma \frac{M}{z-\lambda} = \frac{f(z)}{F(z)} = 0$$

by [13], and consequently also

$$f(z) = 0.$$

It appears, therefore, that the coordinates of every neutral point in our plane give a value of z for which $f(z) = 0$. Moreover for every root of

$f(z) = 0$ there exists a neutral point in our field, since every step that led from [16] to $f(z) = 0$ can be reversed.

The neutral points in question can be determined by various methods. The most obvious method is to scatter iron filings over the horizontal pane which represents the plane of complex numbers. By slightly tapping the pane while the current is flowing, the filings arrange themselves in lines, representing the lines of force. The neutral points or points of equilibrium can then be easily distinguished.

If the non-conducting pane is replaced by a metallic plate which receives the currents for which M is positive and discharges them when M is negative, then the neutral points which we have considered are approximately replaced (the more accurately the larger the plate) by the nodal points of the system of equipotential lines and as such may be located either electrolytically or by means of the galvanometer.

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